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SOLVABILITY OF A CLASS OF DIFFERENTIAL OPERATORS IN $\mathcal{CO}$

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Introduction

Let V and Σ be an involutive submanifold and a lagrangian submanifold of $\sqrt{-1}T^*\mathbb{R}^n$ respectively given as follows:

$$V=\{(x;i\xi)\in\sqrt{-1}T^*\mathbb{R}^n;\xi_1=\cdots=\xi_{n-1}=0\}, \tag{1}$$

$$\Sigma=\{(x;i\xi)\in V;x_n=0\}. \tag{2}$$

In [3], Grigis-Schapira-Sjöstrand obtained a result on the propagation of micro-analyticity of solutions along Σ for transversally elliptic operators P; that is, the principal symbol $\sigma(P)$ of P satisfies

$$|\sigma(P)(x,i\eta)|\sim(|\eta'|+|x_n|\cdot|\eta_n|)^\ell \quad \text{near } \Sigma, \tag{3}$$

where ℓ is some positive integer and $\eta'=(\eta_1,...,\eta_{n-1})$. On the other hand, by using an elementary functorial construction of the sheaf $\widetilde{\mathcal{C}}^2_V$ of small second microfunctions, the first author Funakoshi proved in [2] the solvability of those operators in the space of small second microfunctions as follows:

Theorem 1. *Let $P(x,\partial_x)$ be a differential operator with real analytic coefficcients defined at $x=0$. We suppose that*

$$|\sigma(P)(x,i\eta)|\sim(|\eta'|+|x_n|^k\cdot|\eta_n|)^\ell \quad \textit{near}\,\Sigma \tag{4}$$

for some positive integers k, ℓ. Then we have a sheaf isomorphism:

$$\widetilde{\mathcal{C}}_V^2 \overset{P}{\underset{\sim}{\longrightarrow}} \widetilde{\mathcal{C}}_V^2 \quad \text{on } \overset{\circ}{\pi}^{-1}(\Sigma).$$

Here, $\widetilde{\mathcal{C}}_V^2$ is called the sheaf on $T_V^*\tilde{V}$ of small second microfunctions along V, which satisfies the following exact sequence:

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{C}_{\mathbb{R}^n}|_V \longrightarrow \overset{\circ}{\pi}_* \widetilde{\mathcal{C}}_V^2 \longrightarrow 0, \tag{5}$$

where $\overset{\circ}{\pi} : \overset{\circ}{T}{}_V^* \tilde{V} = T_V^* \tilde{V} \setminus V \to V$ is the canonical projection, $\tilde{V}$ is the patial complexification of V along each leaf of V, and

$$\mathcal{A}_V^2 := \mathcal{C}_{\tilde{V}}|_V = \mathcal{C}_{x_n}\mathcal{O}_{z'}|_V \tag{6}$$

is the sheaf on V of second analytic functions along V. Since any section of $\mathcal{A}_V^2$ has a unique continuation property along each leaf of V, Theorem 1 implies the above-mentioned result of Grigis-Schapira-Sjöstrand. However, to get a solvability result in microfunctions, Theorem 1 is not sufficient. We need a solvability result in $\mathcal{A}_V^2$. Though we have a general result due to Bony and Schapira [1] on solvability in $\mathcal{C}_{\tilde{V}}$ for non-micro-characteristic operators, our operators as in (4) do not fall in such a class of operators.

In this paper, we introduce some special class of differential operators satisfying the property (4), which admit the solvability in $\mathcal{A}_V^2|_\Sigma = \mathcal{C}_{\tilde{V}}|_\Sigma = \mathcal{C}_{x_n}\mathcal{O}_{z'}|_\Sigma$.

Our Main Results

Theorem 2. *Let $P(z, \partial_z)$ be a holomorphic differential operator written in the form:*

$$P(z, \partial_z) = \sum_{|\alpha|+\beta=m} C_{\alpha,\beta} \partial_{z'}^{\alpha} (z_n \partial_{z_n})^{\beta}. \tag{7}$$

Here the $C_{\alpha,\beta}$'s are complex constants satisfying

$$C_{0,m} \neq 0. \tag{8}$$

Then, the morphism

$$P : \mathcal{C}_{x_n} \mathcal{O}_{z'} \to \mathcal{C}_{x_n} \mathcal{O}_{z'}$$

is surjective on $\{x_n = 0\}$.

As a direct corollary of Theorem 1 and Theorem 2, we have

Theorem 3.

Let $P(z, \partial_z)$ be a holomorphic differential operator written as in (7). Suppose that

$$\Big| \sum_{|\alpha|+\beta=m} C_{\alpha,\beta} (\eta')^{\alpha} (x_n)^{\beta} \Big| \sim (|\eta'| + |x_n|)^m \tag{9}$$

for any real small vectors (η', x_n). Then the morphism

$$P : \mathcal{C}_{\mathbb{R}^n} \to \mathcal{C}_{\mathbb{R}^n}$$

is surjective on Σ.

Indeed, condition (9) implies condition (8).

A Sketch of Proof of Theorem 2

Any germ f of $C_{x_n}\mathcal{O}_{z'}$ at $(0,0;idx_n) \in \Sigma$ is written as a boundary value $F(z', x_n + i0)$ of a holomorphic function $F(z)$ in a domain

$$D_r = \{z \in \mathbb{C}^n; |z'| < r, |z_n| < r, \mathrm{Im} z_n > 0\}. \tag{10}$$

Hence, our problem reduces to finding a holomorphic solution U of the following equation for any given $F(z)$ in a complex domain like D_r:

$$P(z, \partial_z)U(z) = F(z). \tag{11}$$

Step 1. Considering the Szegö kernel for a complex ball, we have a decomposition of $F(z)$ for a sufficiently small $r > 0$:

$$F(z) = \text{Const.} \int_{|w'|=r} \frac{F(w', z_n)}{(r^2 - z' \cdot \bar{w}')^n} dS(w'), \tag{12}$$

where $z' \cdot \bar{w}' = \sum_{j=1}^{n-1} z_j \bar{w}_j$ and dS is the surface element. Hence, it is sufficient to solve (11) for any holomorphic function F with continuous parameter w' of the following form:

$$F = G(z' \cdot \bar{w}', z_n; w'), \tag{13}$$

where $G(p, z_n; w')$ is a continuous function on

$$\{(p, z_n, w') \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}; |p| < r, |z_n| < r, \mathrm{Im} z_n > 0, |w'| = r\} \tag{14}$$

depending holomorphically on (p, z_n). Therefore, equation (11) reduces to the following one:

$$C_{0,m} \prod_{j=1}^{m} (z_n \partial_{z_n} - \varphi_j(w')\partial_p) \cdot U(p, z_n; w') = G(p, z_n; w'), \tag{15}$$

where $\{\varphi_j(w'); j = 1, ..., m\}$ are the m-solutions of the algebraic equation

$$\sum_{|\alpha|+\beta=m} C_{\alpha,\beta} \bar{w}'^{\alpha} \phi^{\beta} = 0 \tag{16}$$

in ϕ.

Step 2. By solving first order equations in (15) successively, we get the final solution of (15). Hence, our problem is to solve the following first order equation:

$$(z_n \partial_{z_n} - \varphi_j(w')\partial_p)U(p, z_n; w') = G(p, z_n; w'). \tag{17}$$

In fact, if $\varphi_j(w') \neq 0$, we have a holomorphic solution of (17) of the form

$$U(p, z_n; w') = -\frac{1}{\varphi_j(w')} \int_{\tau(w')}^{p} G(s, z_n e^{(p-s)/\varphi_j(w')}; w') ds. \tag{18}$$

However this solution is not holomorphic in a domain like (14). To get a solution defined in a domain like (14), we must decompose G as

$$G(p, z_n; w') = G_+(p, z_n; w') + G_-(p, z_n; w'). \tag{19}$$

Here, roughly speaking, G_+ is holomorphic in

$$0 < \arg z_n < \pi + \epsilon$$

and G_- is holomorphic in

$$-\epsilon < \arg z_n < \pi$$

for some $\epsilon > 0$. Indeed taking $\tau_\pm(w')$ in the formula (18) as

$$0 < \mp \mathrm{Im}\left(\frac{\tau_\pm(w')}{\varphi_j(w')}\right) < \epsilon \tag{20}$$

respectively, we can show that the corresponding solutions $U_\pm$ are holomorphic in a domain like (14). The most difficult point of our problem is how to treat the case $\varphi_j(w') = 0$. This is not an exceptional problem because for almost all operators P the sets

$$\{w' \in \mathbb{C}^{n-1}; |w'| = 1, \varphi_j(w') = 0 \text{ for some } j\}$$

$$= \{w' \in \mathbb{C}^{n-1}; |w'| = 1, \sum_{|\alpha|=m} C_{\alpha,0} \bar{w'}^{\alpha} = 0\}$$

are not void (but usually of real codimension ≥ 1). To overcome this difficulty, we use a good decomposition of G in (19) based on Hörmander's solution with L^2 -growth order for a $\bar{\partial}$ -equation in the whole $\mathbb{C}$. Before making such a decomposition we choose a better defining function $G(p, z_n : w')$. That is, by solving a Cousin problem on $\mathbb{C} \times \mathbb{P}^1$ with parameter w', we can choose a better defining function $G(p, z_n : w')$, which is holomorphic on

$$\{p \in \mathbb{C}; |p| < r\} \times \{z_n \in \mathbb{P}^1; \mathrm{Im} z_n > 0 \text{ or } |z_n| > r\} \tag{21}$$

satisfying

$$G(p, \infty; w') = 0. \tag{22}$$

Here neglecting variables p, w' we consider a holomorphic function

$$H(\tau) = G(p, e^{\tau}; w') \tag{23}$$

in τ defined on

$$\{\tau \in \mathbb{C}; 0 < \mathrm{Im}\tau < \pi\}$$

with a growth order

$$|H(\tau)| < Ce^{-\mathrm{Re}\tau}$$

as $\mathrm{Re}\tau$ goes to $+\infty$. Now we apply Hörmander's Theorem to the decomposition of H.

Hörmander's Theorem.

Let $\varphi(\tau)$ be a subharmonic function on $\mathbb{C}$. Then for any measurable function $h(\tau)$ satisfying

$$\iint_{\mathbb{C}} |h(\tau)|^2 e^{-\varphi(\tau)} dv(\tau) < \infty \tag{24}$$

we have a weak solution $f(\tau)$ of

$$\frac{\partial}{\partial \bar{\tau}} f(\tau) = h(\tau) \tag{25}$$

satisfying

$$\iint_{\mathbb{C}} |f(\tau)|^2 e^{-\varphi(\tau) - 2\log(|\tau|^2+1)} dv(\tau) < \infty. \tag{26}$$

The detailed proof will be published elsewhere.

References

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